

q -COMPLETE SPACES AND COHOMOLOGY

BY

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1. Introduction. q -complete complex spaces have important cohomological properties. It is not known whether these properties are sufficient to characterize them.

Since in many applications the q -completeness of a space X is of interest mostly because it helps to determine cohomological properties of X , we thought it would be useful to introduce the more general notion of a *cohomologically q -complete space*. The study of these spaces may be an initial step towards a possible characterization of complex spaces with respect to their degree of completeness.

The plan of the paper is as follows: in §2 we give some conditions for q -completeness of a complex space.

In §3 we give the definition of a cohomologically q -complete space and prove some criteria for cohomological q -completeness of a complex space.

In §4 some results in the theory of q -complete spaces are extended to cohomologically q -complete spaces.

In §5 we produce examples of q -complete or cohomologically q -complete spaces and study some of their properties.

We should like to thank Hugo Rossi for a suggestion which led to the proof of Lemma (2.11).

2. q -complete spaces. Let A be an analytic set defined on an open set U of C^N . Let $z=(z_1, \dots, z_N)$ be local coordinates on U . A real-valued function ϕ , defined on A , is said to be *differentiable on A* if there exists on U a differentiable⁽²⁾ function $\hat{\phi}$ such that $\hat{\phi}|_A = \phi$.

The function ϕ is said to be *strongly q -plurisubharmonic* on A , if $\hat{\phi}$ can be chosen so that the Hermitian form in N variables:

$$\mathcal{L}(\hat{\phi}, x) = \sum_{\alpha, \beta=1}^N \left(\frac{\partial^2 \hat{\phi}}{\partial z_\alpha \partial \bar{z}_\beta} \right)_x u_\alpha \bar{u}_\beta$$

has at least $N-q$ positive eigenvalues at every point $x \in U$ ⁽³⁾. $\mathcal{L}(\hat{\phi}, x)$ is called the

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⁽²⁾ By differentiable we always mean differentiable of class C^∞ .

⁽³⁾ There is no uniformity in the literature concerning this terminology. The function ϕ which we call strongly q -plurisubharmonic is sometimes called strongly $(q+1)$ -plurisubharmonic. As a consequence also the complex spaces which we shall call q -complete are sometimes called $(q+1)$ -complete.

Levi form of $\hat{\phi}$ and it will be written $\mathcal{L}(\hat{\phi})$ when there is no possibility of confusion.

Now let X be a reduced complex space (i.e., a complex space in the sense of Serre). A real-valued function ϕ , defined on X , is said to be *differentiable*, respectively *strongly q -plurisubharmonic* on X , if the restrictions of ϕ to a system of coordinate neighborhoods, which determine the given structure of X as a complex space, are differentiable, respectively strongly q -plurisubharmonic.

A complex space X is said to be *q -complete* (see for instance [1, p. 235]) if there exists a differentiable strongly q -plurisubharmonic function ϕ defined on X such that for every $c \in \mathbf{R}$ the sets

$$B_c = \{x \in X \mid \phi(x) < c\}$$

are relatively compact in X .

We recall the following theorem.

(2.1) THEOREM. *A complex space X is 0-complete if and only if it is holomorphically complete (i.e., is a Stein space).*

For the proof of this theorem see [9]. Note that this theorem generalizes to complex spaces the analogous theorem established in [7] for complex manifolds.

The following propositions give useful sufficient conditions for a complex space to be q -complete.

(2.2) PROPOSITION. *Let X be a complex space, q -complete with respect to a function ϕ . Then for every fixed real constant c the open subspace of X :*

$$B_c = \{x \in X \mid \phi(x) < c\}$$

is q -complete.

Proof. We consider on B_c the function $1/(c-\phi)$. For every $s \in \mathbf{R}$ the sets $B_s = \{x \in X \mid 1/(c-\phi(x)) < s\}$ are relatively compact in B_c . In every coordinate neighborhood which contains a point $x \in B_c$, the function $1/(c-\phi)$ is the trace of the function $1/(c-\hat{\phi})$. For every $y \in U$, we compute

$$\mathcal{L}\left(\frac{1}{c-\hat{\phi}}, y\right)(u) = \frac{1}{(c-\hat{\phi})^2} \mathcal{L}(\hat{\phi}, y)(u) + \frac{2}{(c-\hat{\phi})^3} |\text{grad } \hat{\phi} \times u|^2.$$

The last summand, when considered at the points where $\hat{\phi} < c$, is a positive semi-definite form. It follows that the number of positive eigenvalues of $\mathcal{L}(1/(c-\hat{\phi}), y)$ is not less than the number of positive eigenvalues of $\mathcal{L}(\hat{\phi}, y)$. Q.E.D.

(2.3) PROPOSITION. *Let X and Y be two complex spaces. Let X be p -complete and Y be q -complete; then $X \times Y$ is $(p+q)$ -complete.*

Proof. By hypothesis there exists a function $\phi(x)$ strongly p -plurisubharmonic on X such that the sets $B_c(X) = \{x \in X \mid \phi(x) < c\}$ are relatively compact in X . There is also a function $\psi(y)$ strongly q -plurisubharmonic on Y such that the sets $B_c(Y) = \{y \in Y \mid \psi(y) < c\}$ are relatively compact in Y .

The function χ defined on $X \times Y$ by

$$\chi(x, y) = \phi(x) + \psi(y)$$

is clearly strongly $(p+q)$ -plurisubharmonic on $X \times Y$.

Moreover, the sets:

$$B_c(X \times Y) = \{(x, y) \mid \chi(x, y) < c\}$$

are relatively compact in $X \times Y$, as it is easy to check, since the functions ϕ, ψ are bounded from below on X and Y respectively. Q.E.D.

(2.4) PROPOSITION. *Let X, Y be two open subsets of the same complex space. Let X be 0-complete and Y be q -complete. Then $X \cap Y$ is q -complete.*

Proof. Let $\phi(x), \psi(y)$ be the two functions already considered in Proposition (2.3). Define on $X \cap Y$ the function:

$$\xi(x) = \phi(x) + \psi(x), \quad x \in X \cap Y.$$

Let W be a coordinate neighborhood of $x \in X \cap Y$ and let A be an analytic set, defined on an open set $U \subset \mathbb{C}^N$, isomorphic to W . By definition there exist two functions $\hat{\phi}$ and $\hat{\psi}$, respectively, strongly 0- and q -plurisubharmonic on U such that $\hat{\phi}|_A$ coincides with the image of ϕ in A and $\hat{\psi}|_A$ coincides with the image of ψ in A .

The function $\hat{\xi}$ defined on U by

$$\hat{\xi}(z) = \hat{\phi}(z) + \hat{\psi}(z), \quad z \in U,$$

has the property that $\hat{\xi}|_A$ coincides with the image of ξ in A .

By hypothesis there exists a complex linear subspace L of $\mathbb{C}^N$, of complex dimension $N-q$, such that the form $\mathcal{L}(\hat{\psi})(u)$ is positive definite if $u \in L$. Therefore the form $\mathcal{L}(\hat{\xi})(u)$ is positive definite if $u \in L$. This proves that $\mathcal{L}(\hat{\xi})$ has at least $N-q$ positive eigenvalues at every point of U ; thus ξ is strongly q -plurisubharmonic on $X \cap Y$.

Moreover, it is easy to check that the sets:

$$B_c(X \cap Y) = \{x \in X \cap Y \mid \xi(x) < c\}$$

are relatively compact in $X \cap Y$. Q.E.D.

(2.5) REMARK. Proposition (2.4) can be easily generalized. In fact if X is p -complete and Y is q -complete then $X \cap Y$ is $(p+q)$ -complete. Indeed, let L_1, L_2 be two complex linear subspaces of $\mathbb{C}^N$, of dimensions $N-p, N-q$ where the forms $\mathcal{L}(\hat{\phi}), \mathcal{L}(\hat{\psi})$ are positive definite. Then the form $\mathcal{L}(\hat{\xi})$ is positive definite on the intersection $L_1 \cap L_2$ which is a linear subspace of $\mathbb{C}^N$ of dimension $\geq N-(p+q)$.

This result is, in general, of little interest, at least when it happens that

$$p+q \geq \dim_{\mathbb{C}}(X \cap Y).$$

Indeed, if $Z = X \cap Y$ is a manifold, and $n = \dim_{\mathbb{C}} Z$, then the following stronger result holds: Z is n -complete [14].

However, if $p + q < \dim_{\mathbb{C}} (X \cap Y)$, it can be shown that the result of the generalized proposition is the best possible, since $X \cap Y$ may fail to be s -complete for any $s < p + q$ (see §5, Example 5.4).

(2.6) PROPOSITION. *Let X be a 0-complete complex space. Let Y be a subspace of X representable in the form: $Y = \{f_1 = \cdots = f_q = 0\}$ with $f_1, \dots, f_q \in \Gamma(X, \mathcal{O})^{(*)}$. Then the space $X - Y$ is $(q - 1)$ -complete.*

Proof. Let $\phi(x)$ be a strongly 0-plurisubharmonic function on X such that the sets $B_c(X) = \{x \in X \mid \phi(x) < c\}$ are relatively compact in X . For $x \in X - Y$, put:

$$\psi(x) = \frac{1}{\left(\sum_{\alpha=1}^q f_{\alpha}(x) \bar{f}_{\alpha}(x)\right)^q}.$$

We shall prove that the function: $\lambda(x) = \phi(x) + \psi(x)$, $x \in X - Y$, is strongly $(q - 1)$ -plurisubharmonic at every point $x \in X - Y$.

Let $x_0 \in X - Y$; we consider a regular embedding of an open neighborhood $U \subset X - Y$ of x_0 in a suitable open set of $\mathbb{C}^N$. Let $z_1, \dots, z_N$ be coordinates in $\mathbb{C}^N$ and let the embedding be defined by the equations:

$$\begin{aligned} z_1 &= h_1(x) \\ \dots\dots\dots & \\ z_N &= h_N(x) \end{aligned} \quad x \in U,$$

with $h_1, \dots, h_N \in \Gamma(U, \mathcal{O})$.

The neighborhood U is also regularly embedded in an open set $W \subset \mathbb{C}^{N+q}$ by the map $\tau: U \rightarrow \mathbb{C}^{N+q}$ defined by the equations:

$$\begin{aligned} z_1 &= h_1(x), \\ \dots\dots\dots & \\ z_N &= h_N(x), \\ z_{N+1} &= f_1(x), \\ \dots\dots\dots & \\ z_{N+q} &= f_q(x), \end{aligned} \quad x \in U.$$

Since the set

$$\{(z_1, \dots, z_{N+q}) \in \tau(U) \mid z_{N+1} = \cdots = z_{N+q} = 0\}$$

is clearly empty, τ induces a regular embedding of U in

$$W_1 = W - \{z \in W \mid z_{N+1} = \cdots = z_{N+q} = 0\}.$$

(*) $\Gamma(X, \mathcal{O})$ denotes as usual the group of sections of the sheaf $\mathcal{O}$ on X .

Let ϕ_1 be a strongly 0-plurisubharmonic function which extends ϕ to all of W , and let

$$\psi_1 = \frac{1}{\left(\sum_{j=1}^q z_{N+j} \bar{z}_{N+j}\right)^q}.$$

Then the function:

$$\lambda_1(x) = \phi_1(x) + \psi_1(x)$$

extends λ to all of W_1 . We must prove that $\lambda_1(x)$ is strongly $(q-1)$ -plurisubharmonic at every point $z_0 \in W_1$. An easy computation shows that:

$$\frac{\partial^2 \psi_1(x)}{\partial z_{N+\alpha} \partial \bar{z}_{N+\alpha}} = \frac{q \left[(q+1) z_{N+\alpha} \bar{z}_{N+\alpha} - \sum_{j=1}^q z_{N+j} \bar{z}_{N+j} \right]}{\left(\sum_{j=1}^q z_{N+j} \bar{z}_{N+j} \right)^{q+2}}.$$

Let now $z^0 = (z_1^0, \dots, z_N^0, z_{N+1}^0, \dots, z_{N+q}^0) \in W_1$ and let $\alpha_0 \geq 1$ be an index such that:

$$|z_{N+\alpha_0}^0| = \max_{j=1, \dots, q} |z_{N+j}^0|.$$

Then the form $\mathcal{L}(\psi_1, z^0)$ is positive semidefinite on those vectors $(u_1, \dots, u_{N+q})$ whose $q-1$ components $u_{N+1}, \dots, u_{N+\alpha_0-1}, u_{N+\alpha_0+1}, \dots, u_{N+q}$ are zero.

Hence the form $\mathcal{L}(\phi_1 + \psi_1, z^0)$ is positive definite on the same vectors. Since the given vectors form a vector subspace of codimension $q-1$, it follows that $\phi_1 + \psi_1 = \lambda_1$ is strongly $(q-1)$ -plurisubharmonic at the point z^0 .

To complete the proof of Proposition (2.6) it suffices to observe that, by construction, the sets:

$$B_c(X-Y) = \{x \in X-Y \mid \lambda(x) < c\}$$

are relatively compact in $X-Y$. Q.E.D.

(2.7) REMARK. If Y is a complete intersection in X then the integer q which appears in Proposition (2.6) coincides with the complex codimension of Y , with respect to X , at every point of Y . The example (5.1) that we will study in §5 will show that, corresponding to every nonnegative integer j , there exist complex spaces, indeed even manifolds X , holomorphically complete, and analytic subsets Y of codimension q ($q \geq 2$) at every point, such that the manifold $X-Y$ is not $(q+j)$ -complete.

However, if X is a 0-complete manifold, under suitable hypotheses on Y it is still possible to establish a relation between the codimension and the degree of completeness (see Proposition (2.12) below).

We first consider the following situation: let X be a complex manifold, holomorphically complete, of complex dimension n , and let Y be a submanifold of X

of codimension q , regularly embedded in X by means of a finite number (but arbitrarily large) of functions $f_1, \dots, f_r \in \Gamma(X, \mathcal{O})$; i.e., assume

$$Y = \{x \in X \mid f_1(x) = \dots = f_r(x) = 0\}$$

and assume that the Jacobian of $f_1, \dots, f_r$ with respect to the local coordinates of X has rank q at every point of Y . Then we have the following:

(2.8) LEMMA. *Let X and Y satisfy the above hypotheses. Then the manifold $X - Y$ is q -complete.*

Proof. Let ψ be a strongly 0-plurisubharmonic function on X which determines the 0-completeness of X . With no loss of generality we may assume $\psi \geq 0$ on X , since otherwise it would be sufficient to consider the function $\psi + k$ (k suitable real constant) or e^ψ which also determine the 0-completeness of X .

For $m = 1, 2, \dots$ we put:

$$X_m = \{x \in X \mid \psi(x) < m\}$$

and

$$Y_m = X_m \cap Y.$$

Fix an arbitrary point $y \in Y \cap \bar{X}_m$. There exists a coordinate neighborhood U_0 of y in X such that on U_0 the first q coordinates $z_1, \dots, z_q$ are q of the r functions $f_1, \dots, f_r$, while the last $n - q$ coordinates $z_{q+1}, \dots, z_n$ form a system of local coordinates of Y , when restricted to $U_0 \cap Y$. Let y be the origin of this system of coordinates.

For simplicity of notation we shall assume that $z_1 = f_1, \dots, z_q = f_q$. There exists a neighborhood U_1 of y , possibly smaller than U_0 , in which the remaining functions $f_{q+1}, \dots, f_r$ are linear combinations, with holomorphic coefficients, of the functions $z_1, \dots, z_q$:

$$f_{q+j} = z_1 a_{1,j}(z_1, \dots, z_n) + \dots + z_q a_{q,j}(z_1, \dots, z_n) \quad (j = 1, \dots, r - q).$$

We consider on $X - Y$ the function:

$$\phi = -\log \sum_{\alpha=1}^r f_\alpha \bar{f}_\alpha$$

and shall prove that there exists a neighborhood U_2 of y , possibly smaller than U_1 , such that at every point $z^0 = (z_1^0, \dots, z_n^0) \in U_2 - Y$ the restriction of the Levi form $\mathcal{L}(\phi, z^0)$, to the space:

$$z_1 = z_1^0, \dots, z_q = z_q^0,$$

has all its $n - q$ eigenvalues (positive and negative) bounded in absolute value by a suitable constant $K(U_2)$.

Putting $\rho = \sum_{\alpha=1}^r f_{\alpha} \bar{f}_{\alpha}$ one has:

$$\begin{aligned} \frac{\partial^2 \phi}{\partial z_{q+\alpha} \partial \bar{z}_{q+\beta}} &= -\frac{1}{\rho} \sum_{j=1}^{r-q} \left[\left(\sum_{i=1}^q z_i \frac{\partial a_{i,j}}{\partial z_{q+\alpha}} \right) \left(\sum_{i=1}^q \bar{z}_i \frac{\partial \bar{a}_{i,j}}{\partial \bar{z}_{q+\beta}} \right) \right] \\ &\quad + \frac{1}{\rho^2} \left[\sum_{j=1}^{r-q} \left(\sum_{i=1}^q z_i \frac{\partial a_{i,j}}{\partial z_{q+\alpha}} \right) \bar{f}_{q+j} \right] \left[\sum_{j=1}^{r-q} \left(\sum_{i=1}^q \bar{z}_i \frac{\partial \bar{a}_{i,j}}{\partial \bar{z}_{q+\beta}} \right) f_{q+j} \right]. \end{aligned}$$

Take as U_2 any relatively compact subset of U_1 containing y . Then there exists a constant M such that at every point of U_2 one has:

$$\left| \frac{\partial a_{i,j}}{\partial z_{q+\alpha}} \right| < M,$$

for every $i=1, \dots, q$, for every $j=1, \dots, r-q$, and for every $\alpha=1, \dots, r-q$. Moreover, there exists a constant N such that at every point of U_2 ,

$$|a_{i,j}| < N,$$

for every $i=1, \dots, q$ and for every $j=1, \dots, r-q$. It follows that:

$$\begin{aligned} \left| \frac{\partial^2 \phi}{\partial z_{q+\alpha} \partial \bar{z}_{q+\beta}} \right| &\leq \frac{r-q}{\rho} \left[\left(M \sum_{i=1}^q |z_i| \right) \left(M \sum_{i=1}^q |\bar{z}_i| \right) \right] \\ &\quad + \frac{1}{\rho^2} \left[\sum_{j=1}^{r-q} \left(M \sum_{i=1}^q |z_i| \right) |f_{q+j}| \right] \left[\sum_{j=1}^{r-q} \left(M \sum_{i=1}^q |\bar{z}_i| \right) |f_{q+j}| \right] \\ &\leq \frac{M^2}{\rho} (r-q) \left(\sum_{i=1}^q |z_i| \right) \left(\sum_{i=1}^q |\bar{z}_i| \right) \\ &\quad + \frac{M^2}{\rho^2} \left[N(r-q) \left(\sum_{i=1}^q |z_i| \right) \left(\sum_{i=1}^q |\bar{z}_i| \right) \right] \left[N(r-q) \left(\sum_{i=1}^q |z_i| \right) \left(\sum_{i=1}^q |\bar{z}_i| \right) \right]. \end{aligned}$$

On the other hand, $\rho \geq (\sum_{i=1}^q |z_i|)^2 / q^2$ and therefore at every point $z^0 \in U_2 - Y$ one has:

$$\left| \frac{\partial^2 \phi}{\partial z_{q+\alpha} \partial \bar{z}_{q+\beta}} \right| \leq (r-q)q^2 M^2 + (r-q)q^4 M^2 N^2.$$

This proves that all the coefficients of the Levi form $\mathcal{L}(\phi, z^0)$, restricted to the space of the last $n-q$ variables, are bounded on $U_2 - Y$. Hence also the eigenvalues of the same form are bounded on $U_2 - Y$. Therefore there exists, as we claimed, a constant $K(U_2)$ which, at every point $z^0 \in U_2 - Y$, majorizes the absolute values of these eigenvalues.

We now cover $Y \cap \bar{X}_m$ by a finite number of coordinate neighborhoods $U_2^{(i)}$, with the same properties as U_2 and let $K_m = \max_i K(U_2^{(i)})$.

Using the function ψ which determines the 0-completeness of X , we define $\mu(U_2^{(i)}) = \infimum$ of the values taken by the eigenvalues of $\mathcal{L}(\psi, z)$ computed at the points $z \in U_2^{(i)}$.

Let $\mu_m = \min_i \mu(U_2^{(i)})$; μ_m is a positive number.

Let T_m be a constant such that $T_m \mu_m > K_m$ and such that $\mathcal{L}(\phi) + T_m \mathcal{L}(\psi)$ is positive definite on the compact set $\bar{X}_m - \bigcup_i U_2^{(i)}$.

Let $\delta_m = \infimum$ of the values of ϕ on $X_m - Y_m$. Clearly $\delta_m > -\infty$.

Corresponding to each nonnegative integer m we choose now a new constant k_m , such that:

- (i) $k_m \geq T_{m+1}$,
- (ii) $k_m > m + 1 - \delta_{m+2}$,
- (iii) $k_{m+1} \geq k_m$.

We may now use a technique of Andreotti-Narasimhan [2] in order to replace the function ψ by a function $\tilde{\psi}$ such that the function $\phi + \tilde{\psi}$ determines the q -completeness of $X - Y$. To do this we consider a differentiable function $h(t)$, of the real variable t , such that:

$$\begin{aligned} h(t) &= 0 & \text{if } t \leq -1, \\ h(t) &> k_m & \text{if } m \leq t \leq m+1, \\ h'(t) &> 0 & \text{if } t > -1. \end{aligned}$$

Next we define a function $\chi: \mathbf{R} \rightarrow \mathbf{R}$ by:

$$\chi(\lambda) = \int_{-\infty}^{\lambda} h(t) dt$$

and let $\tilde{\psi}(x) = \chi(\psi(x))$.

We shall show that the function $\phi_0 = \phi + \tilde{\psi}$ has the required property. First of all, we shall show that ϕ_0 is strongly q -plurisubharmonic on $X - Y$. To do this, observe that:

$$\mathcal{L}(\phi_0) = \mathcal{L}(\phi) + \mathcal{L}(\tilde{\psi}) \geq \mathcal{L}(\phi) + k_{m-1} \mathcal{L}(\psi)$$

on $(X_m - X_{m-1}) - Y$.

Hence from $k_{m-1} \geq T_m$ it follows that:

$$\mathcal{L}(\phi_0) \geq \mathcal{L}(\phi) + T_m \mathcal{L}(\psi).$$

Therefore $\mathcal{L}(\phi_0)$ has at least $n - q$ positive eigenvalues at every point of $(X_m - X_{m-1}) - Y$. Applying the same reasoning for every m , it follows that ϕ_0 is strongly q -plurisubharmonic at every point of $X - Y$.

In order to prove that $X - Y$ is q -complete with respect to ϕ_0 it remains to show that the sets:

$$B_c^0 = \{x \in X - Y \mid \phi_0(x) < c\}$$

are relatively compact in $X - Y$ for all $c \in \mathbf{R}$. Actually it will suffice to prove this statement for integral values of c . Assume then that c is an integer. We shall prove

that $\bar{B}_c^0$ (closure of B_c^0 in X) has empty intersection with Y and that $\bar{B}_c^0$ is contained in the compact set $\bar{X}_c$. Let y be a point of Y . There exists an open neighborhood $U \subset X$ of y such that $\phi(x) > c - \tilde{\psi}(x)$ for all $x \in U - Y$ (we may choose U to be relatively compact in X ; then $\tilde{\psi}(x)$ is bounded from below on U). Hence no point of U belongs to B_c^0 , so $y \notin \bar{B}_c^0$, and this proves precisely that $\bar{B}_c^0 \cap Y = \emptyset$.

Finally, $\bar{B}_c^0$ is contained in $\bar{X}_c$, for if $x \notin X_c - Y$, then there exists a nonnegative integer s such that $x \in (X_{c+s+1} - X_{c+s}) - Y$.

Then one has:

$$\psi(x) \geq c + s$$

whence

$$\tilde{\psi}(x) \geq k_{c+s-1} > c + s - \delta_{c+s+1} \geq c + s - \phi(x)$$

so that

$$\phi_0(x) = \phi(x) + \tilde{\psi}(x) > c + s \geq c$$

and this implies that $x \notin \bar{B}_c^0$.

Thus $X - Y$ is q -complete with respect to ϕ_0 . Q.E.D.

(2.9) LEMMA. *Let Y be a submanifold of codimension q , regularly embedded in X ; $\dim_{\mathbb{C}} X = n$. Then there exists a regular embedding of Y in X by a finite number of functions of $\Gamma(X, \mathcal{O})$.*

Proof. The submanifold Y , as a set, can be represented as the zeroes of a finite number of global holomorphic functions $f_1, \dots, f_k$. (Actually k could be taken equal to n [6].)

Let $\mathcal{I}$ be the sheaf of ideals of Y in X . Let F be the Fréchet space direct sum of q copies of $\Gamma(X, \mathcal{I})$. Then an element of F is a q -tuple of global holomorphic functions on X , vanishing on Y .

Let $y \in Y$. Let F_y be the subset of F of those q -tuples $(f_1, \dots, f_q) \in F$ whose Jacobian with respect to a system of local coordinates $z_1, \dots, z_n$ of X near y , has rank q at y .

For the proof we need the fact that the set $F - F_y$ is of the first category in F ⁽⁵⁾ as we shall prove in the next lemma.

Let N_0 be a countable subset everywhere dense in Y . Then, $\bigcup_{y \in N_0} (F - F_y) = F - \bigcap_{y \in N_0} F_y$ is a set of the first category in F , but F is not of the first category in F , hence $\bigcap_{y \in N_0} F_y$ is not empty. Therefore there exists a q -tuple

$$f_{0,1}, \dots, f_{0,q} \in \Gamma(X, \mathcal{I})$$

such that the Jacobian of $f_{0,1}, \dots, f_{0,q}$ with respect to the chosen system of local coordinates of X near y , has rank q at every point $y_0 \in N_0$.

⁽⁵⁾ A set N is said to be of the first category if $N = \bigcup_{\alpha=1,2,\dots} N_\alpha$ with $(N_\alpha)^0 = \emptyset$.

Now Y , as a set, may also be represented as:

$$Y = \{x \in X \mid f_1 = \cdots = f_k = f_{0,1} = \cdots = f_{0,q} = 0\};$$

moreover the embedding defined by these functions is regular outside a closed analytic subset $Y_0 \subset Y$ which is nowhere dense in Y . Hence $\dim_C Y_0 < \dim_C Y$.

Now let N_1 be a countable subset everywhere dense in Y_0 . By the same argument, there exist functions $f_{1,1}, \dots, f_{1,q} \in \Gamma(X, \mathcal{J})$, $(f_{1,1}, \dots, f_{1,q}) \in \bigcap_{y \in N_1} F_y$. Furthermore:

$$Y = \{x \in X \mid f_1 = \cdots = f_k = f_{0,1} = \cdots = f_{0,q} = f_{1,1} = \cdots = f_{1,q} = 0\}$$

and the embedding defined by these functions is regular outside a closed analytic subset $Y_1 \subset Y_0$ with $\dim_C Y_1 < \dim_C Y_0$.

After at most $n-q+1$ steps one has:

$$(2.10) \quad Y = \{x \in X \mid f_1 = \cdots = f_k = f_{0,1} = \cdots = f_{0,q} = \cdots = f_{n-q,1} = \cdots = f_{n-q,q} = 0\}$$

and these functions define a regular embedding of Y in X . Q.E.D.

It remains to prove the following lemma.

(2.11) LEMMA. $F - F_y$ is a set of the first category in F .

Proof. It suffices to prove that F_y is open and everywhere dense in F .

First of all we shall prove that F_y is not empty. In fact, since the embedding of Y in X is regular, an open set $U \subset X$, $y \in U$, and a q -tuple $(\phi_1, \dots, \phi_q) \in \Gamma(U, \mathcal{J})$ exist such that:

$$\text{rank} \left(\frac{\partial(\phi_1, \dots, \phi_q)}{\partial(z_1, \dots, z_n)} \right)_y = q.$$

Now from Theorem A (Cartan-Serre) it follows that there exist functions $h_1, \dots, h_d \in \Gamma(X, \mathcal{J})$ such that:

$$\phi_\alpha = \sum_{\beta=1}^d a_{\alpha\beta} h_\beta \quad (\alpha = 1, \dots, q)$$

with $a_{\alpha\beta} \in \Gamma(V, \mathcal{O})$ where V is an open set containing y , $V \subset U$.

Moreover, putting $A_y = (a_{\alpha\beta}(y))$, one has:

$$\left(\frac{\partial(\phi_1, \dots, \phi_q)}{\partial(z_1, \dots, z_n)} \right)_y = A_y \left(\frac{\partial(h_1, \dots, h_d)}{\partial(z_1, \dots, z_n)} \right)_y.$$

Hence there are functions $h_{i_1}, \dots, h_{i_q}$, among the $h_1, \dots, h_d$ such that:

$$\text{rank} \left(\frac{\partial(h_{i_1}, \dots, h_{i_q})}{\partial(z_1, \dots, z_n)} \right)_y = q,$$

which shows that F_y is not empty.

Next, F_y is clearly open in F .

Finally, we prove that F_y is everywhere dense in F . With no loss of generality we may assume that:

$$\text{rank} \left(\frac{\partial(h_1, \dots, h_q)}{\partial(z_1, \dots, z_q)} \right)_y = q.$$

Let $(g_1, \dots, g_q) \in F$. We consider $(g_1 + \lambda h_1, \dots, g_q + \lambda h_q) \in F$ where λ is any complex number, and prove that for sufficiently small $\lambda \neq 0$ the rank of

$$J_y(\lambda) = \left(\frac{\partial(g_1 + \lambda h_1, \dots, g_q + \lambda h_q)}{\partial(z_1, \dots, z_q)} \right)_y$$

is equal to q . This will prove density.

In fact $\det J_y(\lambda)$ is a polynomial of degree q in λ which does not vanish identically, since the coefficient of λ^q is

$$\det \left(\frac{\partial(h_1, \dots, h_q)}{\partial(z_1, \dots, z_q)} \right)_y,$$

which is different from zero.

Therefore all the values of λ which are not roots of the equation $\det J_y(\lambda) = 0$ give elements of F_y . Q.E.D.

Lemma (2.9) proves that the hypotheses of Lemma (2.8) are always satisfied for any regular embedding of Y in X . Hence:

(2.12) PROPOSITION. *Let X be a holomorphically complete manifold. Let Y be a regularly embedded submanifold of X of codimension q . Then the manifold $X - Y$ is q -complete.*

As a consequence of Lemma (2.9) we have the following proposition which we wish to note because of its intrinsic interest.

(2.13) PROPOSITION. *Let X be a holomorphically complete manifold. Let Y be a regularly embedded submanifold of X and let $\mathcal{I}$ be the sheaf of ideals of Y in X . Then $\Gamma(X, \mathcal{I})$ is a finitely generated $\Gamma(X, \mathcal{O})$ -module.*

Proof. We denote by $s_1, \dots, s_p$ the functions defining the regular embedding of Y in X in (2.10). The sequence:

$$0 \longrightarrow \text{Ker } \alpha \longrightarrow \mathcal{O}^p \xrightarrow{\alpha} \mathcal{I} \longrightarrow 0$$

where $\alpha(v_i) = s_i$, v_i being the i th unit vector in $\mathcal{O}^p$, is exact. We thus have the exact sequence:

$$\dots \longrightarrow \Gamma(X, \mathcal{O}^p) \xrightarrow{\alpha_*} \Gamma(X, \mathcal{I}) \longrightarrow H^1(X, \text{Ker } \alpha) \longrightarrow \dots$$

Since $\text{Ker } \alpha$ is a coherent analytic sheaf, it follows from Theorem B (Cartan-Serre) that $H^1(X, \text{Ker } \alpha) = 0$. Hence the homomorphism α_* is surjective. Q.E.D.

3. Cohomologically q -complete spaces. An important result in the theory of q -complete spaces is expressed by the following theorem.

(3.1) THEOREM [1, p. 250]. *Let X be a q -complete complex space, let $\mathcal{F}$ be any coherent analytic sheaf on X . Then*

$$(3.2) \quad H^i(X, \mathcal{F}) = 0 \quad \text{for } i \geq q+1.$$

(3.3) DEFINITION. *A complex space X will be called cohomologically q -complete if for every coherent analytic sheaf $\mathcal{F}$ on X the property (3.2) holds.*

A cohomologically q -complete space is clearly also cohomologically $(q+1)$ -complete.

(3.4) THEOREM [10]. *A complex space of complex dimension n which is countable at infinity is cohomologically n -complete.*

Clearly a q -complete space is cohomologically q -complete by Theorem (3.1). J.-P. Serre has shown that the converse is true when $q=0$:

(3.5) THEOREM. *A complex space X , which is cohomologically 0-complete is necessarily 0-complete.*

The proof is divided into two parts. First, one proves that every cohomologically 0-complete space is holomorphically complete. Indeed, when X is a manifold, this result can be found in [5, p. 53]; and a similar proof can be given in the general case. The theorem now follows from Theorem (2.1).

(3.6) PROPOSITION. *Let X be a complex space. Let $X = \bigcup_{m=1,2,\dots} B_m$, where $\{B_m\}$ is an increasing sequence of open subsets of X . If each B_m is cohomologically q -complete, then X is cohomologically $(q+1)$ -complete.*

For the proof of this proposition, see [4]. Some similar results are also proved in [1] (see in particular §20).

(3.7) PROPOSITION. *Let X be a complex space; let $X = X_1 \cup X_2$, with X_1, X_2 open subspaces of X such that for some fixed integer $q > 0$:*

- (i) X_1, X_2 are cohomologically q -complete,
- (ii) $X_1 \cap X_2$ is cohomologically $(q-1)$ -complete. Then X is cohomologically q -complete.

Proof. Let $\mathcal{F}$ be a coherent analytic sheaf on X . We shall denote by $\mathcal{F}$ also the restrictions $\mathcal{F}|_{X_1}, \mathcal{F}|_{X_2}, \mathcal{F}|_{X_1 \cap X_2}$. They too are coherent analytic sheaves. Then the following Mayer-Vietoris sequence holds:

$$\cdots \rightarrow H^{i-1}(X_1 \cap X_2, \mathcal{F}) \rightarrow H^i(X, \mathcal{F}) \rightarrow H^i(X_1, \mathcal{F}) \oplus H^i(X_2, \mathcal{F}) \rightarrow \cdots$$

Hence from hypotheses (i) and (ii) it follows:

$$0 \rightarrow H^i(X, \mathcal{F}) \rightarrow 0 \quad \text{if } i \geq q+1. \quad \text{Q.E.D.}$$

The latter proposition has the following generalization:

(3.8) PROPOSITION. *Let X be a complex space. Let $X = \bigcup_{j=1}^h X_j$, with X_j open subspaces such that:*

(i) X_j is cohomologically q -complete ($j=1, \dots, h$),

(ii) $(X_1 \cup \dots \cup X_j) \cap X_{j+1}$ is cohomologically $(q-1)$ -complete ($j=1, \dots, h-1$).

Then X is cohomologically q -complete.

Proof. Applying Proposition (3.7) to the triple $(X_1 \cup X_2, X_1, X_2)$ it follows that $X_1 \cup X_2$ is cohomologically q -complete. Therefore Proposition (3.7) may be applied to triple $(\bigcup_{j=1}^3 X_j, \bigcup_{j=1}^2 X_j, X_3)$. After $h-1$ steps the proof is complete.

Q.E.D.

4. Some properties of cohomologically q -complete manifolds. Theorem 2 of [12] can be extended to cohomologically q -complete manifolds. Namely, one has:

(4.1) THEOREM. *Let X be a cohomologically q -complete manifold. Then every complex-valued differential form, d -closed, of degree $n+q$ on X is cohomologous to a differential form of type (n, q) , $\bar{\partial}$ -closed (and therefore d -closed).*

Putting:

$$(4.2) \quad H^{p,q}(X, C) = \{\xi \in H^{p+q}(X, C) \mid \xi \text{ can be represented in the de Rham isomorphism by a form of type } (p, q)\},$$

Theorem (4.1) shows that if X is cohomologically q -complete, then:

$$H^{n+q}(X, C) \cong H^{n,q}(X, C).$$

The proof is the same as in [12] since only cohomological properties are used.

(4.3) REMARK. Let X_0 be a complex manifold of complex dimension n , and let $X \subset X_0$ be an open submanifold such that $\bar{X} = X_0$.

The problem of finding a representation of a holomorphic function on X_0 by an integral over an $(n+q)$ -dimensional cycle of X , ($0 \leq q \leq n-1$), is connected with the study of the cohomology group $H^{n+q}(X, C)$. On the other hand, if ϕ^{n+q} represents, via de Rham's theorem, an element of $H^{n+q}(X, C)$, the condition $d\phi^{n+q} = 0$ does not necessarily imply that $d(f\phi^{n+q}) = 0$ for every $f \in \Gamma(X_0, \mathcal{O})$.

But if ϕ^{n+q} is of type (n, q) then certainly $d(f\phi^{n+q}) = 0$ for every $f \in \Gamma(X_0, \mathcal{O})$. Therefore the problem is ultimately reduced to the study of the group $H^{n,q}(X, C)$.

These two aspects of the problem are related under the hypotheses of Theorem (4.1), for if X is q -complete, then the cohomology class of ϕ^{n+q} always contains a differential form of type (n, q) .

With respect to the homology of cohomologically q -complete manifolds one has the following result, similar to the corollary of [11, p. 304]:

(4.4) THEOREM. *Let X be a cohomologically q -complete manifold of complex dimension n . Then:*

$$H_{n+i}(X, C) = 0 \quad \text{for } i \geq q+1.$$

Proof. One has:

$$H^i(X, \Omega^n) = 0 \quad \text{if } i \geq q+1.$$

This means, by the Dolbeault isomorphism, that every differential form $\phi^{n,i}$ of type (n, i) , $\bar{\partial}$ -closed, is of the form:

$$\phi^{n,i} = \bar{\partial}\psi^{n,i-1}.$$

On the other hand, if $\phi^{n,i}$ is any d -closed differential form of type (n, i) , $\phi^{n,i}$ is also $\bar{\partial}$ -closed. Furthermore, $\phi^{n,i} = \bar{\partial}\psi^{n,i-1}$ implies that:

$$\phi^{n,i} = d\psi^{n,i-1}.$$

This equality shows that the groups $H^{n,i}(X, C)$, defined in (4.2), are zero. Hence from the isomorphism proved in Theorem (4.1) it follows that:

$$H^{n+i}(X, C) = 0 \quad \text{for } i \geq q+1.$$

Therefore also $\text{Hom}(H_{n+i}(X, C), C) = H^{n+i}(X, C) = 0$. This implies that $H_{n+i}(X, C) = 0$ for $i \geq q+1$. Q.E.D.

5. Examples.

(5.1) We consider in C^{2n} ($n \geq 2$), the two linear complex subspaces of dimension n : $C^n(z_1, \dots, z_n)$ given by the equations $z_{n+1} = \dots = z_{2n} = 0$, $C^n(z_{n+1}, \dots, z_{2n})$ given by the equations $z_1 = \dots = z_n = 0$. We put:

$$X_1 = C^{2n} - C^n(z_1, \dots, z_n), \quad X_2 = C^{2n} - C^n(z_{n+1}, \dots, z_{2n}),$$

$$X = X_1 \cup X_2, \quad Y = C^n(z_1, \dots, z_n) \cup C^n(z_{n+1}, \dots, z_{2n}).$$

Then one has:

$$X = C^{2n} - \{0\}, \quad X_1 \cap X_2 = C^{2n} - Y.$$

Let $\mathcal{F}$ be a coherent analytic sheaf on X . Since X_1 and X_2 are $(n-1)$ -complete (see Proposition (2.6)), it follows from the Mayer-Vietoris sequence that:

$$0 \rightarrow H^i(X_1 \cap X_2, \mathcal{F}) \rightarrow H^{i+1}(X, \mathcal{F}) \rightarrow 0 \quad \text{for } i \geq n.$$

Thus we have the isomorphisms:

$$(5.2) \quad H^i(X_1 \cap X_2, \mathcal{F}) \cong H^{i+1}(X, \mathcal{F}) \quad (i \geq n).$$

Now take $\mathcal{F} = \Omega^{2n}$, the sheaf of germs of holomorphic forms of maximum degree over X . We shall prove that $H^{2n-1}(X, \Omega^{2n}) \neq 0$.

Indeed, X is $(2n-1)$ -complete; hence from Theorem (4.1) it follows that:

$$H^{2n, 2n-1}(X, C) \cong H^{4n-1}(X, C).$$

Since $X = C^{2n} - \{0\}$ is contractible to the sphere S^{4n-1} , then

$$H^{4n-1}(X, C) \cong H^{4n-1}(S^{4n-1}, C) \cong C \neq 0.$$

Now $H^{2n,2n-1}(X, \mathbb{C}) \cong \mathbb{C}$ in a natural way, a quotient of $H^{2n-1}(X, \Omega^{2n})$ and thus it follows that $H^{2n-1}(X, \Omega^{2n}) \neq 0$.

Hence from the isomorphism (5.2), for $i=2n-2$, it follows that:

$$H^{2n-2}(X_1 \cap X_2, \Omega^{2n}) \neq 0.$$

This proves that *for any nonnegative integer $j < n-2$, the space obtained by removing the subset Y of complex codimension n from $\mathbb{C}^{2n}$ is not $(n+j)$ -complete*. This example shows that the hypothesis that Y be regularly embedded in X can not be removed in Proposition (2.12).

With a more subtle argument, it can be proved (see for instance [3]), that $H^{2n-1}(X, \Omega^{2n})$ is a complex vector space of infinite dimension; hence from the isomorphism (5.2), it follows that $\dim_{\mathbb{C}} H^{2n-2}(X_1 \cap X_2, \Omega^{2n}) = +\infty$. This proves that the space $\mathbb{C}^{2n} - Y$ not only is not q -complete, for $q < 2n-2$, but it is not even *strongly q -pseudoconvex*⁽⁹⁾ for $q < 2n-2$, as one could have conjectured.

(5.3) REMARK. In example (5.1), Y has codimension $q \geq 2$ at each of its points. It is not possible to give similar examples with $q=1$. In fact, if X is a 0-complete manifold and one removes from X any analytic subset Y of codimension 1 at each of its points, then the manifold $X - Y$ is 0-complete.

(5.4) Example (5.1) may be modified in the following way: consider in $\mathbb{C}^{n+m}$ ($n, m \geq 2$), the two linear complex subspaces $\mathbb{C}^n(z_1, \dots, z_n)$ represented by the equations $z_{n+1} = \dots = z_{n+m} = 0$ and $\mathbb{C}^m(z_{n+1}, \dots, z_{n+m})$, by the equations $z_1 = \dots = z_n = 0$. Put:

$$X_1 = \mathbb{C}^{n+m} - \mathbb{C}^n(z_1, \dots, z_n), \quad X_2 = \mathbb{C}^{n+m} - \mathbb{C}^m(z_{n+1}, \dots, z_{n+m}),$$

$$X = X_1 \cup X_2, \quad Y = \mathbb{C}^n(z_1, \dots, z_n) \cup \mathbb{C}^m(z_{n+1}, \dots, z_{n+m}).$$

By the same argument as in example (5.1) one proves that, for any integer $j < n+m-2$, the space $\mathbb{C}^{n+m} - Y$ is not j -complete.

Noting that $\mathbb{C}^{n+m} - Y = X_1 \cap X_2$ and that X_1, X_2 are $(m-1)$ - and $(n-1)$ -complete, this example proves that the result of (2.5) can not be improved.

(5.5) In $\mathbb{C}^n$ ($n > 1$), consider the two following domains:

$$X_1 = \{(z_1, \dots, z_n) \mid 1/2 < |z_1| < 1, |z_2|^2 + \dots + |z_n|^2 < 1\},$$

$$X_2 = \{(z_1, \dots, z_n) \mid |z_1| < 1, |z_2|^2 + \dots + |z_n|^2 < 1/2\}.$$

The domain $X = X_1 \cup X_2$ is *cohomologically 1-complete*. In fact, consider the domain $X_1 \cap X_2$ defined by:

$$X_1 \cap X_2 = \{(z_1, \dots, z_n) \mid 1/2 < |z_1| < 1, |z_2|^2 + \dots + |z_n|^2 < 1/2\}.$$

Then it is clear that $X_1, X_2, X_1 \cap X_2$ are holomorphically complete and therefore 0-complete. Hence from Proposition (3.7), it follows that X is cohomologically 1-complete.

⁽⁹⁾ For the definition of a strongly q -pseudoconvex space see, for instance, [1].

(5.6) REMARK. The domain X of the preceding example occurs in some proofs of Hartog's theorem. This theorem shows, among other things, that the domain X is not holomorphically complete, since every holomorphic function on X can be continued outside X .

(5.7) Finally consider in C^n the linear complex hyperplanes C_j^{n-1} defined by the equation $z_j=0$. For a fixed integer q ($q=0, 1, \dots, n-1$), consider the following domains:

$$\begin{aligned} X_1 &= C^n - C_1^{n-1}, \\ X_2 &= C^n - C_2^{n-1}, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X_q &= C^n - C_q^{n-1}, \\ X_{q+1} &= C^n - \bigcup_{j=q+1}^n C_j^{n-1}. \end{aligned}$$

We shall show that the domain $X = \bigcup_{j=1}^{q+1} X_j$ is q -complete. In fact one has $X = C^n - A$, where:

$$A = \{z_1 = 0, \dots, z_q = 0, z_{q+1} \cdot \dots \cdot z_n = 0\}.$$

Since A is represented as locus of zeroes of $q+1$ global holomorphic functions, it follows from Proposition (2.6) that X is q -complete.

(5.8) REMARK. The domain X in example (5.7) is q -complete; moreover, $\dim_C H^{n+q}(X, C) = 1$. According to Remark (4.3), this guarantees the existence of differential forms $\phi^{n,q}$ on X such that every function f , holomorphic on C^n , can be evaluated at the origin by means of the integral of $f\phi^{n,q}$ over an $(n+q)$ -dimensional cycle of X .

The explicit determination of the forms $\phi^{n,q}$, together with the study of the general topological aspects of the cycles to be used in the integration was carried out by E. Martinelli [8], for a domain Y which is the intersection of a finite number of domains of the same type as the domain X considered above. For such a domain Y , one has $\dim_C H^{n+q}(Y, C) > 1$; the resulting integral formulae are symmetric in the n variables involved [15].

A similar study was subsequently carried out in [13] for the domain X itself.

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